

ON THE HOWSON PROPERTY OF DESCENDING HNN-EXTENSIONS OF GROUPS

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ABSTRACT. A group G is said to have the Howson property (or to be a Howson group) if the intersection of any two finitely generated subgroups of G is finitely generated subgroup. It is proved that descending HNN-extension is not a Howson group under some assumptions satisfied by the base group of HNN-extension. In particular, a result of the paper joined with a Burns – Brunner result (received in 1979) implies that any descending HNN-extension of non-cyclic free group does not have the Howson property.

1. Introduction. Main results

A group G is said to have the Howson property (or to be a Howson group) if the intersection of any two finitely generated subgroups of G is the finitely generated subgroup too. This denomination was introduced into practice after the work of A. G. Howson [1], where it was proved that any free group possesses this property. Then, generalizing this result, B. Baumslag [2] have shown that a free product of two Howson groups is a Howson group. On the other hand, it was noted in [3] that the direct product of free group of rank 2 and of infinite cyclic group does not have the Howson property. This observation was then extended by R. Burns and A. Brunner: they have proved in [4] that any extension of non-cyclic finitely generated free group by infinite cyclic group is not a Howson group. Since every extension by infinite cyclic group is splittable, any such group is a special case of descending HNN-extension of free group.

Recall that descending (or named by some authors as ascending) HNN-extension is, in turn, a special case of general construction of HNN-extension and can be defined as follows.

Let G be a group and let φ be an injective endomorphism of G . Then descending HNN-extension of (base) group G with respect to endomorphism φ is the group $G(\varphi) = (G, t; t^{-1}gt = g\varphi \ (g \in G))$ generated by generators of G and by one more element t and defined by all defining relations of G and by all relations of form $t^{-1}gt = g\varphi$ where $g \in G$. It is obvious that if endomorphism φ is, in addition, surjective (i. e. it is an automorphism of G) then the group $G(\varphi)$ turns out to be a splitting extension of group G by infinite cyclic group with generator t . Therefore, the following assertion can be considered as a supplement to the Burns – Brunner above result:

Theorem 1. *Let G be non-cyclic finitely generated free group and let φ be an injective but not surjective endomorphism of G . Then the descending HNN-extension $G(\varphi) = (G, t; t^{-1}gt = g\varphi \ (g \in G))$ is not a Howson group.*

Thus, this result joined with the Burns – Brunner result implies that *any descending HNN-extension of non-cyclic free group does not have the Howson property*.

The assumption that the base group of the HNN-extension is non-cyclic is essential. Indeed, any HNN-extension of infinite cyclic group is an one-relator group $G_k = \langle a, t; t^{-1}at = a^k \rangle$ (where k is a non-zero integer) belonging to the family of Baumslag – Solitar groups, and it was shown in [3] that all G_k are Howson groups. It is relevant to mention that this result was generalized in [4] as follows:

The group $G = \langle a_1, a_2, \dots, a_m, t; t^{-1}ut = v \rangle$, where u and v are non-identity elements of free group $F = \langle a_1, a_2, \dots, a_m \rangle$, is Howson group provided that at least one of u and v is not a proper power in F .

One more family of one-relator Howson groups provides the result of work [5] asserting that the generalized free product of two free groups with cyclic amalgamated subgroup which is isolated at least in one of free factors is a Howson group.

On the other hand, many one-relator groups do not possess the Howson property. It was shown in [3] that if non-abelian one-relator group with non-trivial center is not isomorphic to group $G_{-1} = \langle a, t; t^{-1}at = a^{-1} \rangle$ then it is not a Howson group. It should be noted that this assertion turns out to be a consequence of the Burns – Brunner result since non-cyclic one-relator group with non-trivial center is an extension of non-cyclic finitely generated free group by infinite cyclic group [6]. Recently some new examples of one-relator groups without Howson property were given in [7], [8] and [9]. However, it is easy to see that all these groups are a descending HNN-extensions of non-cyclic free group. Thus, the impracticability of Howson property in all examples of one-relator non-Howson groups that we know up to now is in fact a consequence of our Theorem 1 and Burns – Brunner result.

Theorem 1 is a special case of the following somewhat more general result. Let us say that a subgroup H of group G is freely complemented if there exists a non-identity subgroup K of G such that subgroup generated by subgroups H and K is their free product $H * K$.

Theorem 2. *Let G be a finitely generated group, let φ be an injective but not surjective endomorphism of G and $H = G\varphi$. If subgroup H of group G is freely complemented then the descending HNN-extension $G(\varphi)$ is not a Howson group.*

In order to deduce Theorem 1 from Theorem 2 it is enough to note that if G is a non-cyclic finitely generated free group and φ is an injective but not surjective endomorphism of G then subgroup $H = G\varphi$ is freely complemented. In fact, since rank of subgroup H is equal to rank of G and H is a proper subgroup of G , the Schreier's formula implies that H is of infinite index in G . Therefore, it follows from the Hall – Burns Theorem (see e. g. [10, proposition 1.3.10]) that H is freely complemented.

One more application of Theorem 2 is

Corollary. *Let a finitely generated group G is the free product of non-identity groups A B . If φ is an injective but not surjective endomorphism of G such that $A\varphi \subseteq A$ and $B\varphi \subseteq B$ then $G(\varphi)$ is not a Howson group.*

In this case subgroup $H = G\varphi$ is generated by subgroups $A\varphi$ $B\varphi$ (and is a free product of them) and since $H\varphi \neq G$ then $A\varphi \neq A$ or $B\varphi \neq B$. Therefore, H is of infinite index in G and hence (see e. g. [11, p. 27]) subgroup H is freely complemented.

The similar assertion is fulfilled for group that is decomposable into the direct product:

Theorem 3. *Let group G be a direct product of non-identity groups A and B and let φ be an injective but not surjective endomorphism of G such that $A\varphi \subseteq A$ and $B\varphi \subseteq B$. If $A\varphi \neq A$, $B\varphi \neq B$ and at least one of subgroups A and B is finitely generated then $G(\varphi)$ is not a Howson group.*

Theorem 3 implies, in particular, that if G is a free abelian finitely generated group and φ is injective endomorphism of G such that the matrix of φ in some free base of G is of block-diagonal form where determinant of at least two diagonal blocks is not equal to ± 1 then $G(\varphi)$ is not a Howson group. The problem of complete characterization of those descending HNN-extensions of free abelian groups that are a Howson groups is still open.

2. The proof of Theorem 2

Let φ be an injective but not surjective endomorphism of finitely generated group G , let $H = G\varphi$ and K be a non-identity subgroup of G such that subgroup L generated by subgroups H and K is their free product, $L = H * K$. It is obvious that we can assume subgroup K to be finitely generated.

For any integer n let $K_n = t^{-n}Kt^n$. Let also N denote the subgroup of group $G(\varphi)$ that is generated by all subgroups K_n and M denote the subgroup of group $G(\varphi)$ that is generated by all subgroups K_n with $n \geq 0$. Remark that for $n \geq 0$ we have $K_n = K\varphi^n$ and therefore subgroup M is contained in the base group G of HNN-extension $G(\varphi)$.

Lemma 1. *Subgroup N is the free product of family subgroups K_n , $n \in \mathbb{Z}$. Hence subgroups N and M are not finitely generated.*

In order to prove Lemma 1 it is enough to prove that any subgroup generated by a finite family of subgroups K_n is the free product of these subgroups, and to this end, in turn, it is enough to prove that for any integer $r \geq 1$ subgroup M_r generated by subgroups $K_0 = K\varphi^0$, $K_1 = K\varphi$, \dots , $K_r = K\varphi^r$ is the free product of these subgroups.

When $r = 1$ this is obvious since $L = H * K = H * K_0$ and $K\varphi \leq H$. Let us assume that for some $r \geq 1$ subgroup M_r is the free product of subgroups K_0, K_1, \dots, K_r . Then since the mapping φ is an isomorphism of group G on the group H and for any $i \geq 0$ $K_i\varphi = K_{i+1}$ subgroup $M_r\varphi$ is the free product of subgroups K_1, K_2, \dots, K_{r+1} . Since subgroup M_{r+1} is generated by subgroups K_0 and $M_r\varphi$ and $M_r\varphi \leq H$ this implies that subgroup M_{r+1} is the free product of subgroups K_0, K_1, \dots, K_{r+1} . The proof of Lemma 1 is complete.

Lemma 2. $N \cap G = M$.

Since the inclusion $M \subseteq N \cap G$ is trivial it is enough to prove the opposite inclusion. Any non-identity element u of subgroup N can be written in the form

$$u = v_1 v_2 \cdots v_r,$$

where $r \geq 1$, for any $i = 1, 2, \dots, r$ v_i is non-identity element from some subgroup K_{n_i} , $v_i = t^{-n_i} g_i t^{n_i}$ for some non-identity element $g_i \in K$, and if $r > 1$ then for any $i = 1, 2, \dots, r-1$ $n_i \neq n_{i+1}$.

We shall show that if at least one of the numbers n_1, n_2, \dots, n_r is negative, then element u does not enter in subgroup G . Since otherwise the inclusion $u \in M$ is evident by that the proof of Lemma will be complete.

So, let us suppose that for some i , $1 \leq i \leq r$, we have $n_i < 0$. If $r = 1$ then since element g_1 does not belong to subgroup H , the expression $u = t^{-n_1} g_1 t^{n_1}$ is reduced in HNN -extension $G(\varphi)$ and therefore $u \notin G$ by Britton's Lemma.

Now, let $r > 1$ and n denote the smallest from integers n_1, n_2, \dots, n_r . Suppose by the contrary that element u belongs to subgroup G . Then since $n \leq -1$ element $t^n u t^{-n} = u \varphi^{-n}$ belongs to subgroup H .

On the other hand since $n - n_i \leq 0$ for any $i = 1, 2, \dots, r$, we have for every such number i

$$t^n v_i t^{-n} = t^{n-n_i} g_i t^{-(n-n_i)} = g_i \varphi^{n_i-n} \in K \varphi^{n_i-n}.$$

Therefore, since for any $i = 1, 2, \dots, r-1$ integers $n_i - n$ and $n_{i+1} - n$ are different, the following expression of element $t^n u t^{-n}$,

$$t^{-n} u t^n = g_1 \varphi^{n_1-n} \cdot g_2 \varphi^{n_2-n} \cdot \dots \cdot g_r \varphi^{n_r-n},$$

is reduced in decomposition of group M into free product in Lemma 1.

By the choice of integer n there exists at least one number i such that $n_i - n = 0$; let $i_1 < i_2 < \dots < i_s$ be all numbers of those syllables $g_i \varphi^{n_i-n}$ for which this equality is satisfied. The rest syllables in this expression of element $t^n u t^{-n}$ belong to subgroup H and by join all such consecutive syllables we obtain the expression of element $t^n u t^{-n}$ of form

$$t^n u t^{-n} = w_0 g_{i_1} w_1 g_{i_2} w_2 \dots w_{s-1} g_{i_s} w_s,$$

where all w_j are elements of subgroup H that are not equal to identity except for, may be, w_0 and w_s . In any case this expression is reduced in free decomposition $L = H * K$ of subgroup L and since at least one syllable of it belongs to subgroup K , this contradicts to inclusion $t^n u t^{-n} \in H$. Lemma 2 is proved.

Now we can complete the proof of Theorem 2. Let F be subgroup of group $G(\varphi)$ generated by subgroup K and element t . We shall show that $F \cap G = M$. Since subgroups F and G are finitely generated while subgroup M (by Lemma 1) is not finitely generated, this will imply that the group $G(\varphi)$ is not a Howson group.

Arbitrary element $f \in F$ can be written in the form

$$f = g_0 t^{n_1} g_1 t^{n_2} \dots t^{n_r} g_r$$

where g_0, g_1, \dots, g_r are some elements from subgroup K and n_1, n_2, \dots, n_r are some integers. The factorization of group $G(\varphi)$ by the normal closure of subgroup G shows evidently that if element f belongs to subgroup G then $n_1 + n_2 + \dots + n_r = 0$ and therefore $f \in N$. Thus, we have inclusion $F \cap G \subseteq N$ and this with taking into account of Lemma 2 and obvious inclusion $M \subseteq F$ implies that

$$F \cap G = F \cap G \cap N = F \cap M = M.$$

3. The proof of Theorem 3

Let $G = A \times B$ and let φ be an injective endomorphism of group G such that $A\varphi \subseteq A$ and $B\varphi \subseteq B$. Suppose also that $A\varphi \neq A$, $B\varphi \neq B$ and subgroup A is finitely generated. Let the restriction of mapping φ on subgroup B be denoted by φ too and let $B(\varphi) = (B, t; t^{-1}bt = b\varphi \ (b \in B))$ be corresponding descending HNN-extension of group B .

It is easy to see that there exists a homomorphism ρ of group $G(\varphi)$ to the group $B(\varphi)$ which sends the stable letter of group $G(\varphi)$ onto stable letter of group $B(\varphi)$ and action of which on subgroup G coincides with action of projection $\pi : G \rightarrow B$. We claim that the kernel of ρ is equal to subgroup $U = \bigcup_{k=0}^{\infty} t^k A t^{-k}$.

Indeed, since $A\rho = A\pi = 1$ the inclusion $U \subseteq \text{Ker } \rho$ is evident. Backwards, arbitrary element v from $\text{Ker } \rho$ (just as any element of group $G(\varphi)$) can be written in form $v = t^m g t^{-n}$ for some integers $m \geq 0$ and $n \geq 0$ and some element $g \in G$. Let $g = ab$ where $a \in A$ and $b \in B$. Then $u\rho = t^m b t^{-n}$ and therefore in group $B(\varphi)$ we have the equality $t^m b t^{-n} = 1$. Since in any HNN-extension the stable letter generates subgroup that intersects the base group trivially then $b = 1$ and $m = n$. Thus, $v = t^m a t^{-m} \in U$ and the proof of equality $\text{Ker } \rho = U$ is complete.

Remark that since $A\varphi \neq A$ subgroup U is the union of strictly increasing sequence of subgroups and therefore is not finitely generated.

Let C denote subgroup of group $G(\varphi)$ generated by subgroup A and element t and let D denote subgroup generated by subgroup A and element tb where $b \in B \setminus B\varphi$.

It is evident that $U \leq C$ and it is easy to see that subgroup U also is contained in D . In fact, we have $A \leq D$. If for some $k \geq 0$ subgroup $t^k A t^{-k}$ is contained in D then D contains subgroup $(tb) t^k A t^{-k} (tb)^{-1}$. But since $bt^k = t^k b\varphi^k$ we have $(tb) t^k A t^{-k} (tb)^{-1} = t^{k+1} A t^{-(k+1)}$.

Thus, subgroup U is contained in intersection of subgroups $C \cap D$. We shall prove now that, in fact, $C \cap D = U$. Since subgroups C and D are finitely generated and subgroup U is not finitely generated then the proof of Theorem 3 will be complete.

The image of subgroup C under homomorphism ρ of group $G(\varphi)$ on group $B(\varphi)$ is the cyclic subgroup generated by element t and the image of subgroup D is the cyclic subgroup generated by element tb . If the intersection $C\rho \cap D\rho$ of these subgroups would be non-trivial then for some non-zero integers m and n in group $B(\varphi)$ must be fulfilled the equation $t^m = (tb)^n$. The passage to the quotient of group $B(\varphi)$ by normal closure of subgroup B shows that $m = n$. Consequently, in group $B(\varphi)$ the equation $t^m = (tb)^m$ is fulfilled, where the integer m may be supposed to be positive. Since

$$(tb)^m = t^m \cdot t^{-(m-1)} b t^{m-1} t^{-(m-2)} b t^{m-2} \dots t^{-1} b t b = t^m \cdot b\varphi^{m-1} b\varphi^{m-2} \dots b\varphi b,$$

we have the equality $b\varphi^{m-1} b\varphi^{m-2} \dots b\varphi b = 1$. This implies the inclusion $b \in B\varphi$ which contradicts to the choice of element b .

So, $C\rho \cap D\rho = 1$ and therefore $(C \cap D)\rho = 1$. Since the kernel of ρ coincides with subgroup U this implies the required inclusion $C \cap D \subseteq U$.

References

1. Howson A. G., On the intersection of finitely generated free groups, J. London Math. Soc. 29. 1954. 428–434.
2. Baumslag B., Intersections of finitely generated subgroups in free products, J. London Math. Soc. 41. 1966. 673–679.

3. Moldavanskii D. I., On the intersection of finitely generated subgroups, Siberian Math. J. 1968, 9, 1422–1426 (Russian).
4. Burns R., Brunner A., Two remarks on group Howson property, Algebra and Logic, 18, 5 (1979), 513–522 (Russian).
5. Burns R., On finitely generated subgroups of an amalgamated product of two subgroups, Trans. Amer. Math. Soc. 169 (1972), 293–306.
6. Baumslag G., Taylor T., The center of groups with one defining relator, Math. Ann. 1968, Vol. 175, 315–319.
7. Kapovich I., Howson property and one-relator groups, Commun in algebra, 1999, Vol. 27, 1057–1072.
8. Bezverhnyaya N. B., On the Howson property and hyperbolicity of some 2-generated 1-related groups, Algorithmic problems in groups and semigroups, Tula, 2001 (Russian).
9. Bezverhnyaya N. B. On the Howson property of some one-relator groups, Chebyshevskii sbornik (Publishers of Tula State Pedagogic University), Vol. 2 (2001), 14–18 (Russian).
10. Lyndon R. C., Schupp P. E., Combinatorial group theory, Springer - Verlag, Berlin, etc. 1977.
11. Goryushkin A. P., Goryushkin V. A. Free decomposability and balance of groups, Publishers of Kamchatka State Pedagogic University, 2005.(Russian)

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